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FORMULA SHEET

# Exam P

SOA / CAS · Probability

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FORMULAS

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TOPICS

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## PROBABILITY FUNDAMENTALS

5 items

### Combinations (unordered)

$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$

$n$ objects taken $r$ at a time, order does not matter

### Addition rule (three events)

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

### Addition rule (two events)

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

### Permutations (ordered)

${}_nP_r = \frac{n!}{(n-r)!}$

$n$ objects taken $r$ at a time, order matters

### Set complement

$P(A^c) = 1 - P(A)$

## CONDITIONAL PROBABILITY & BAYES

5 items

### Independence of two events

$A$ and $B$ are independent iff $P(A \cap B) = P(A)\,P(B)$

(equivalently: $P(A \mid B) = P(A)$)

### Conditional probability

$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0$

### Law of total probability

$P(A) = \sum_i P(A \mid B_i)\,P(B_i)$

where $\{B_i\}$ is a partition of the sample space

### Bayes' theorem

$P(B_i \mid A) = \frac{P(A \mid B_i)\,P(B_i)}{\sum_j P(A \mid B_j)\,P(B_j)}$

### Multiplication rule

$P(A \cap B) = P(A \mid B)\,P(B) = P(B \mid A)\,P(A)$

## DISCRETE DISTRIBUTIONS

6 items

### Poisson distribution — PMF, mean, variance

$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$

$E[X] = \lambda, \quad \mathrm{Var}(X) = \lambda$

$k = 0, 1, 2, \ldots$

### Negative Binomial distribution — PMF, mean, variance

$P(X = k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}$ (trials for $r$th success)

$E[X] = \frac{r}{p}, \quad \mathrm{Var}(X) = \frac{r(1-p)}{p^2}$

### Binomial distribution — PMF, mean, variance

$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$

$E[X] = np, \quad \mathrm{Var}(X) = np(1-p)$

$k = 0, 1, \ldots, n$

### Bernoulli distribution — PMF, mean, variance

$P(X = 1) = p,\ P(X = 0) = 1 - p$

$E[X] = p, \quad \mathrm{Var}(X) = p(1-p)$

### Hypergeometric distribution — PMF, mean, variance

$P(X = k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$

$E[X] = \frac{nK}{N}, \quad \mathrm{Var}(X) = \frac{nK(N-K)(N-n)}{N^2(N-1)}$

### Geometric distribution — PMF, mean, variance

$P(X = k) = (1-p)^{k-1} p$ (number of trials to first success)

$E[X] = \frac{1}{p}, \quad \mathrm{Var}(X) = \frac{1-p}{p^2}$

$k = 1, 2, \ldots$

## CONTINUOUS DISTRIBUTIONS

8 items

### Normal distribution — PDF

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

$E[X] = \mu, \quad \mathrm{Var}(X) = \sigma^2$

### Memoryless property (Exponential / Geometric)

$P(X > s + t \mid X > s) = P(X > t)$

Only the Exponential (continuous) and Geometric (discrete) satisfy this.

### Exponential distribution — PDF, CDF, mean, variance

$f(x) = \lambda e^{-\lambda x}, \quad F(x) = 1 - e^{-\lambda x}, \quad x > 0$

$E[X] = \frac{1}{\lambda}, \quad \mathrm{Var}(X) = \frac{1}{\lambda^2}$

### Uniform distribution — PDF, mean, variance

$f(x) = \frac{1}{b-a}, \quad a < x < b$

$E[X] = \frac{a+b}{2}, \quad \mathrm{Var}(X) = \frac{(b-a)^2}{12}$

### Weibull distribution — PDF, mean

$f(x) = \frac{\tau}{\theta}\left(\frac{x}{\theta}\right)^{\tau-1} e^{-(x/\theta)^\tau}, \quad x > 0$

$E[X] = \theta\,\Gamma\left(1 + \frac{1}{\tau}\right)$

### Gamma distribution — PDF, mean, variance

$f(x) = \frac{x^{\alpha-1}e^{-x/\theta}}{\Gamma(\alpha)\,\theta^\alpha}, \quad x > 0$

$E[X] = \alpha\theta, \quad \mathrm{Var}(X) = \alpha\theta^2$

$\alpha$=shape, $\theta$=scale

### Lognormal distribution — PDF, mean, variance

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right), \quad x > 0$$

$E[X] = e^{\mu+\sigma^2/2}, \quad \mathrm{Var}(X) = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$

### Beta distribution — PDF, mean

$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}, \quad 0 < x < 1$

$E[X] = \frac{\alpha}{\alpha+\beta}, \quad \mathrm{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

## JOINT & MARGINAL DISTRIBUTIONS

3 items

### Conditional PDF

$f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$

### Joint PDF — marginal densities

$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$

$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$

### Independence of continuous RVs

$X, Y$ independent iff $f_{X,Y}(x,y) = f_X(x)\,f_Y(y)$ for all $x, y$

## EXPECTATION & VARIANCE

7 items

### Covariance

$\mathrm{Cov}(X,Y) = E[XY] - E[X]\,E[Y]$

If $X, Y$ independent: $\mathrm{Cov}(X,Y) = 0$

### MGF definition and moment extraction

$M_X(t) = E[e^{tX}]$

$E[X^n] = M_X^{(n)}(0)$ ($n$th derivative at $t = 0$)

### Correlation coefficient

$\rho_{XY} = \frac{\mathrm{Cov}(X,Y)}{\sigma_X\,\sigma_Y}, \quad -1 \le \rho \le 1$

### Variance of a linear combination

$\mathrm{Var}(aX + bY) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y) + 2ab\,\mathrm{Cov}(X,Y)$

### Variance shortcut

$\mathrm{Var}(X) = E[X^2] - (E[X])^2$

### Law of total expectation

$E[X] = E[E[X \mid Y]]$

### Law of total variance

$\mathrm{Var}(X) = E[\mathrm{Var}(X \mid Y)] + \mathrm{Var}(E[X \mid Y])$

### Ordinary deductible — payment per loss

$Y^L = \max(X - d,\, 0)$

$E[Y^L] = E[X] - E[X \wedge d]$

where $d$ = deductible, $X$ = ground-up loss

### Stop-loss (aggregate) premium

$E[(S - d)_+] = E[S] - E[S \wedge d]$

$S$ = aggregate loss, $d$ = retention (aggregate deductible)

### Limited expected value (LEV)

$E[X \wedge u] = \int_0^u S(x)\,dx = \int_0^u [1 - F(x)]\,dx$

for $X \geq 0$

### Payment per payment (excess loss)

$e(d) = E[X - d \mid X > d] = \dfrac{E[X] - E[X \wedge d]}{1 - F(d)}$

$d$ = deductible

### Policy limit — payment per loss

$Y^L = \min(X,\, u) = X \wedge u$

$E[Y^L] = E[X \wedge u]$

$u$ = policy limit